POLITECNICO
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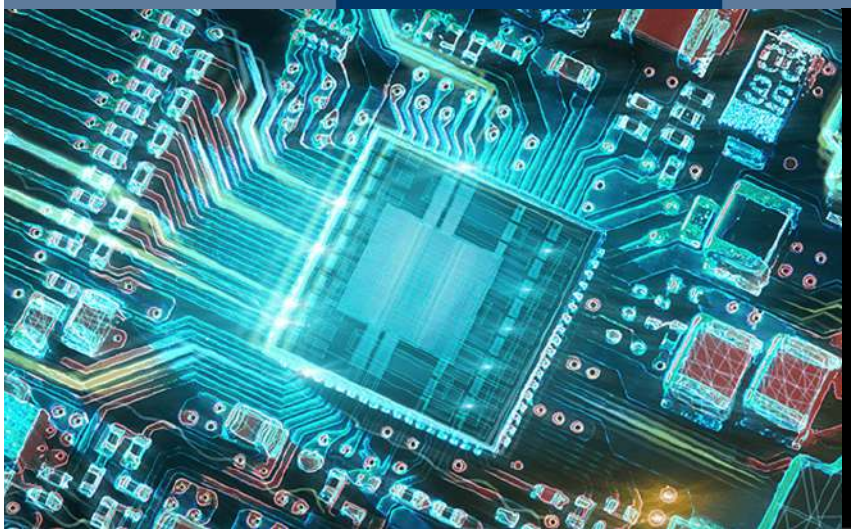
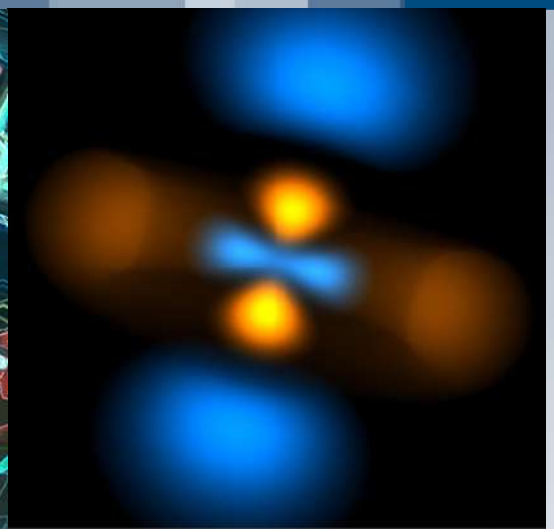
Quantum bits, gates and circuits





Sommario

- Introduzione
- Quantum bits
- Quantum gates
- Quantum circuits
- Conclusioni

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Introduction

- These slides serve as a final seminar of the course ‘Solid State Electronics’
- Objectives:
  - Understand the fundamental concepts of quantum computing
  - Assess the state of the art of quantum technology
  - Highlight the value of quantum mechanics for developing novel computing systems
- Some material is taken from the literature (review papers, tutorial slides, see references) and from the following text book: R. LaPierre, «Introduction to Quantum Computing,», Springer Nature Switzerland AG 2021

Sommario

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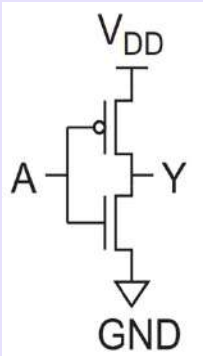
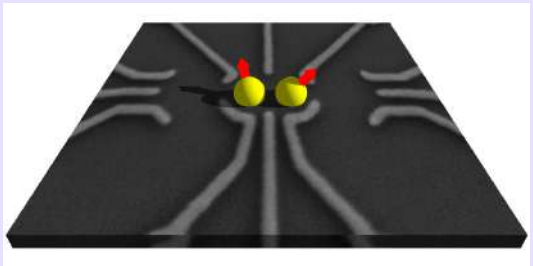
Quantum computing

- **Bit**: **B**inary dig**it** di informazione dell'algebra Booleana
- Può essere 0 o 1
- Rappresentato dal potenziale al nodo Y nell'inverter CMOS



$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

- **Qubit** (quantum bit)= bit di informazione $|\psi\rangle$ nel quantum computing (QC)
- **Dirac notation** per indicare che è quantum
- Si trova in una **sovrapposizione** di 0 e 1
- Rappresentato dallo spin di un elettrone



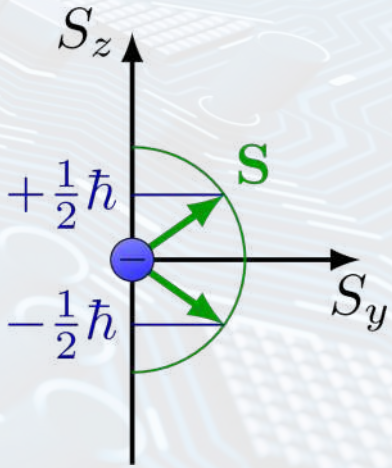
Example: the electron spin

- An electron can be seen as a spinning ball rotating on its axis
- The modulus of the spin angular momentum is quantized:

$$S^2 = \hbar s(s+1)$$

- The z-component of the spin angular momentum is also quantized:

$$S_z = s_z \hbar \qquad s_z = -s, -s+1, \dots, s-1, s$$



- Two types of particles:
  - <u>Bosons</u> with integer s (e.g., photons has s = 1, α has s = 0)
  - <u>Fermions</u> with half-integer s (e.g., electrons, neutrons and protons have s = ½)
- Electrons have 2 spin basis states, namely $S_z = \hbar/2$ or $S_z = -\hbar/2$

The spin eigenfunction

- According to quantum mechanics, spin is described by a wavefunction:

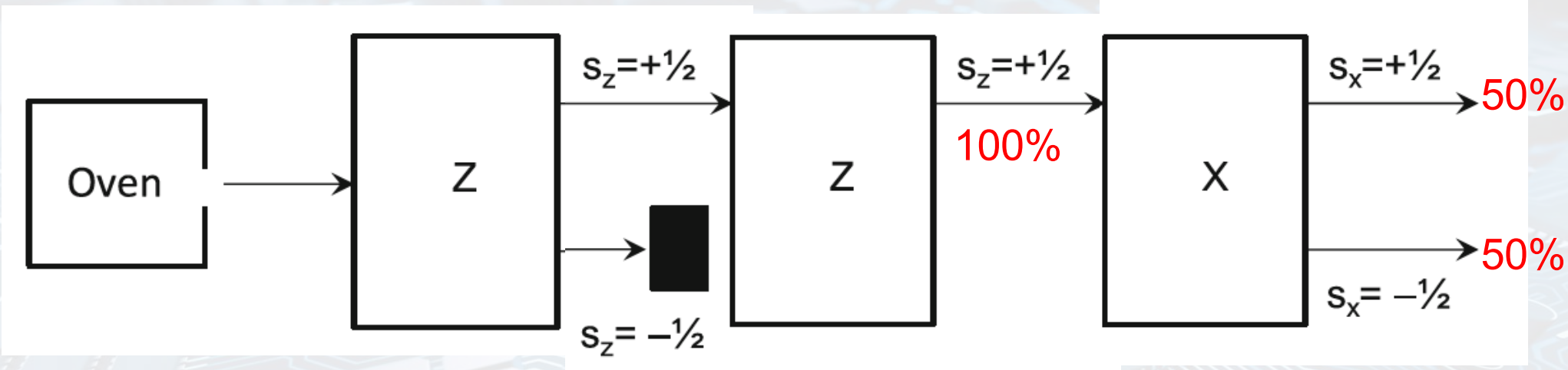
$$|\psi\rangle = \alpha|z_+\rangle + \beta|z_-\rangle$$

  - Both α and β are complex numbers with $|\alpha|^2 + |\beta|^2$ to guarantee normalization
  - Basis states = $|z_+\rangle$ and $|z_-\rangle$, also called $|0\rangle$ and $|1\rangle$,
  - Superposition state = $|\psi\rangle$
- When measuring the spin:
  - $|\alpha|^2$ gives the probability of finding spin up $|z_+\rangle$
  - $|\beta|^2$ gives the probability of finding spin up $|z_-\rangle$
- Similarly, we can introduce spin states wavefunction along other directions according to:

$$|x_+\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle + |z_-\rangle) \qquad |y_+\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle + \mathrm{i}|z_-\rangle)$$

$$|x_-\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - |z_-\rangle) \qquad |y_-\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - \mathrm{i}|z_-\rangle)$$

Measurement: the Stern-Gerlach experiment



- A SG apparatus along z-axis will split electrons with spin up (s_z = ½) and spin down (s_z = -½) with equal probabilities = 50%
- Second measurement yields 100% spin up → state s_z = ½ has no component along s_z = -½ state (s_z = ½ and s_z = -½ are <u>orthogonal states</u>)
- Third measurement: equal splitting of state s_x = +½ and state s_x = -½ , therefore s_z = +½ can be seen as an equal superposition of state s_x = +½ and state s_x = -½

Vector representation

- State $|\psi\rangle = \alpha|z_+\rangle + \beta|z_-\rangle$ can be represented as a 2x1 vector: $|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$
- Thus we can write:

$$|z_+\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |x_+\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle + |z_-\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad |y_+\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle + \mathrm{i}|z_-\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$|z_-\rangle = \begin{pmatrix}0\\1\end{pmatrix} \qquad |x_-\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - |z_-\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \qquad |y_-\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - \mathrm{i}|z_-\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

Spin operators

- In quantum mechanics, any observable is associated with an operator
- Any operator obeys an equation $\hat{O}|\psi\rangle$ =o$|\psi\rangle$ where o is the eigenvalue and $|\psi\rangle$ is the eigenfunction
- Physical interpretation: when measuring observable O on eigenstate (or basis state) $|\psi\rangle$, one obtains eigenvalue o
- Thus we can introduce spin operators for each spin component, namely:

$$\hat{S}_x|x_+\rangle = \frac{\hbar}{2}|x_+\rangle \qquad \hat{S}_y|y_+\rangle = \frac{\hbar}{2}|y_+\rangle \qquad \hat{S}_z|z_+\rangle = \frac{\hbar}{2}|z_+\rangle$$

$$\hat{S}_x|x_-\rangle = -\frac{\hbar}{2}|x_-\rangle \qquad \hat{S}_y|y_-\rangle = -\frac{\hbar}{2}|y_-\rangle \qquad \hat{S}_z|z_-\rangle = -\frac{\hbar}{2}|z_-\rangle$$

- Pauli spin operators $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ are defined as: $\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x$ etc.

Spin operators in matrix form

- Since states can be represented by vectors, operators take the form of matrices, e.g.:

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_x|x_+\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}|x_+\rangle$$

$$\hat{S}_x|x_-\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}|x_-\rangle$$

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{S}_y|y_+\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{2}|y_+\rangle$$

$$\hat{S}_y|y_-\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ i \end{pmatrix} = -\frac{\hbar}{2}|y_-\rangle$$

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_z|z_+\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}|z_+\rangle$$

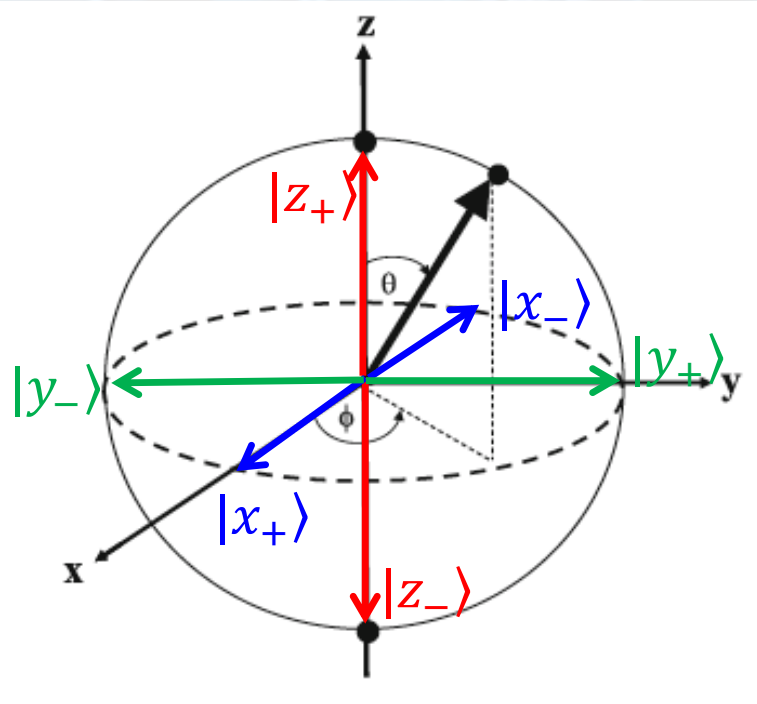
$$\hat{S}_z|z_-\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\frac{\hbar}{2}|z_-\rangle$$

Bloch sphere

- It is possible to show that a generic spin state $|n_+\rangle$ can be expressed as:

$$|\psi\rangle = cos\frac{\theta}{2}|z_+\rangle + e^{i\phi} sin\frac{\theta}{2}|z_-\rangle$$

- Which is a point on the Bloch sphere
- Notable points:
  - $|z_+\rangle$ with $\theta = 0$ (North pole)
  - $|z_-\rangle$ with $\theta = \pi$ (South pole)
  - $|x_+\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle + |z_-\rangle)$ with $\theta = \frac{\pi}{2}$ and $\phi = 0$
  - $|x_-\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - |z_-\rangle)$ with $\theta = \frac{\pi}{2}$ and $\phi = \pi$
  - $|y_+\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle + i|z_-\rangle)$ with $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$
  - $|y_-\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle - i|z_-\rangle)$ with $\theta = \frac{\pi}{2}$ and $\phi = \frac{3\pi}{2}$



Summary of qubit

- Qubit = Quantum bit representing a quantum state:

$$|\psi\rangle = \alpha|z_+\rangle + \beta|z_-\rangle = \alpha|1\rangle + \beta|0\rangle$$

- within a basis of two orthogonal states (eigenstates of a given operator)
- Examples of qubits:
  - Spin state of a single electron (spin up $|z_+\rangle$ or down $|z_-\rangle$)
  - Energy state of a confined electron (ground or excited state)
  - Direction of the current within a superconducting circuit
  - Path of a single photon (path 1 or path 2)
  - Polarization of a single photon (horizontal or vertical)
- Coefficients must obey normalization: $\alpha^2 + \beta^2 = 1$
- A qubit can simultaneously represent 2 states and 2 complex numbers (α, β)
- n qubits can represent all the available 2^n states and 2^n complex numbers

Double qubit

- Given qubits 1 and 2, the 2-qubit state can be written as:

$$|\psi\rangle = (\alpha_1|0\rangle_1 + \beta_1|1\rangle_1)(\alpha_2|0\rangle_2 + \beta_2|1\rangle_2)$$

$$|\psi\rangle = \alpha_1\alpha_2|0\rangle_1|0\rangle_2 + \alpha_1\beta_2|0\rangle_1|1\rangle_2 + \beta_1\alpha_2|1\rangle_1|0\rangle_2 + \beta_1\beta_2|1\rangle_1|1\rangle_2$$

- Which can be rewritten as:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

- Vectorial form given by: $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$
- With normalization condition:

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$$

- Note: not all 2-qubit states can be factored as a product of one-qubit states (see entanglements)

Triple qubit

- Similarly one can form 3-qubit states:

$$|\psi\rangle = \alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \ldots + \alpha_{111}|111\rangle = \begin{pmatrix} \alpha_{000} \\ \alpha_{001} \\ \alpha_{010} \\ \alpha_{011} \\ \alpha_{100} \\ \alpha_{101} \\ \alpha_{110} \\ \alpha_{111} \end{pmatrix}$$

- The number of coefficients increase exponentially:
  - 1 qubit = 2 parameters (α and β)
  - 2-qubits = 2^2 parameters (α_{00}, α_{01}, α_{10} and α_{11})
  - 3 qubits = 2^3 parameters (α_{000}, α_{001}, …)
  - 300 qubits = 2^{300} parameters = 10^{90} = number of particles in the known universe
- Quantum computing: superposition allows an exponential increase of states → high level of parallelism that is at the origin of the quantum computing speedup

Potenza computazionale del quantum computing

Multi-bit: esempio 3 bit possono essere configurati in 8 = 2^3 stati:

b2	b1	b0
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Multi-qubit = 3 qubit sono descritti da uno stato $|\psi\rangle$ dato dalla sovrapposizione di 2^3 = 8 stati base:

$$
\begin{aligned}
|\psi\rangle = \; & c_0|000\rangle + \\
& c_1|001\rangle + \\
& c_2|010\rangle + \\
& c_3|011\rangle + \\
& c_4|100\rangle + \\
& c_5|101\rangle + \\
& c_6|110\rangle + \\
& c_7|111\rangle
\end{aligned}
$$

Servono 2^N numeri per descrivere N qubits

La legge di crescita esponenziale

Progressione esponenziale della capacità di un sistema di qubit:

- 40 qubits → $2^{40} = 10^{12}$ = 1 TByte
- 50 qubits → $2^{50} = 10^{15}$ = 1 PByte
- 76 qubits → $2^{76} = 10^{23}$ = 100 Zbyte = come tutti i dati finora generati
- …
- 300 qubits → $2^{300} = 10^{90}$ = come il numero di tutte le particelle dell'universo

Roadmap of IBM quantum technology

Entanglement

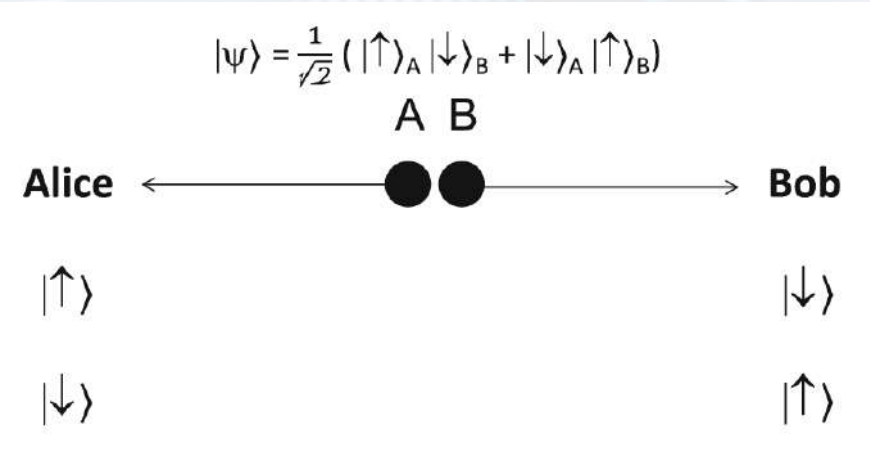
- Entangled states cannot be factored as $|\psi\rangle = (\alpha_1|0\rangle_1 + \beta_1|1\rangle_1)(\alpha_2|0\rangle_2 + \beta_2|1\rangle_2)$
- Example:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

- Measuring $|\psi_1\rangle$ gives 50% probability of $|0\rangle$ and 50% probability of $|1\rangle$
- Measuring $|\psi_2\rangle$ gives 50% probability of $|0\rangle$ and 50% probability of $|1\rangle$
- However, measuring $|\psi_1\rangle$ in state $|0\rangle$ forces $|\psi_2\rangle$ to be in state $|0\rangle$ → the measurement of $|\psi_1\rangle$ dictates the state of $|\psi_2\rangle$
- After the measurement, state $|00\rangle$ (or $|11\rangle$) becomes a separable state

Bell states and EPR pairs

- Notable entangled states are the Bell states:



$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

- In the Einstein-Podolsky-Rosen (EPR) experiment, a pair of electrons (A,B) is prepared in a Bell state, e.g. $|\Psi^+\rangle$, then A and B are sent to different positions
- If Alice measures state $|0\rangle$ for A, then B will instantaneously collapse to $|1\rangle$
- Information is sent faster than light? No, since a random measurement is no 'information' → EPR is not a paradox

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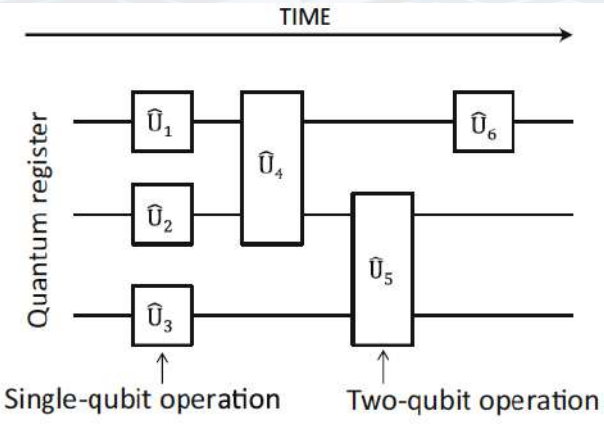
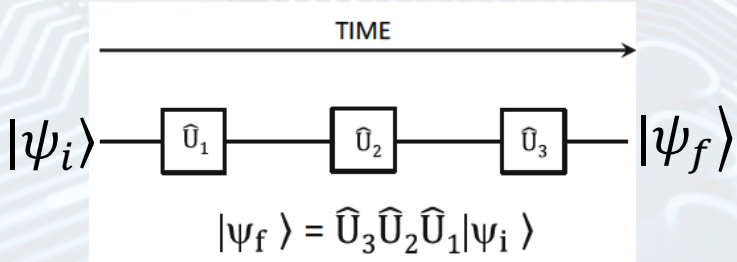
Quantum gates

- A quantum gate can be seen as an operator transforming a qubit from an initial state to a final state:

$$\hat{U}|\psi_i\rangle = |\psi_f\rangle$$

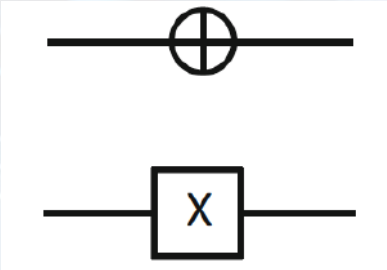
- For a single qubit, $|\psi_i\rangle$ and $|\psi_f\rangle$ are 2x1 vectors, thus $\hat{U}$ is a 2x2 matrix
- For two qubits, states are 4x1 vectors, thus $\hat{U}$ is a 4x4 matrix
- Quantum gates are represented by blocks such as:





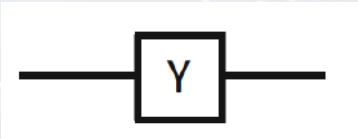
Pauli gates

- Single qubit gate is defined by operator $U = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ where $U\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$
- Notable gates: Pauli gates
- X gate: $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$
- Matrix form: $X = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$



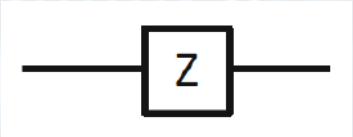
- Bit flip or NOT
- π rotation around the x-axis in the Bloch sphere

- Y gate: $Y|0\rangle = i|1\rangle$ and $Y|1\rangle = -i|0\rangle$
- Matrix form: $Y = i|1\rangle\langle 0| - i|0\rangle\langle 1| = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y$



- Phase shift + bit flip
- π rotation around the y-axis in the Bloch sphere

- Z gate: $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$
- Matrix form $Z = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z$



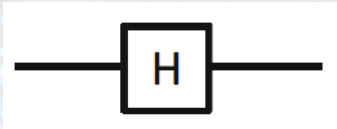
- Phase shift
- π rotation around the z-axis in the Bloch sphere

- Note: if applied twice, X, Y and Z all result in identity I (the 4th Pauli gate)
- https://lewisla.gitbook.io/learning-quantum/quantum-circuits/single-qubit-gates

Hadamard gate

- Hadamard gate: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

  $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$

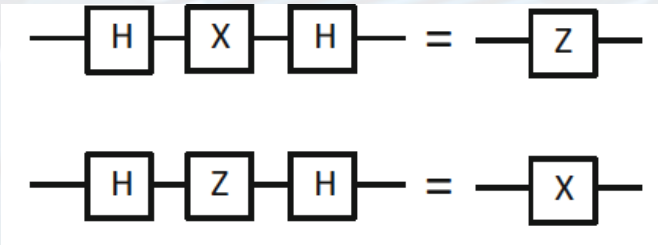


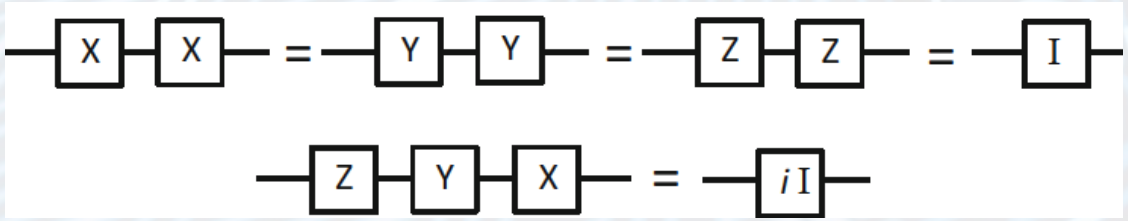
- Transform z-basis into x-basis
- π rotation around the axis $(\hat{x} + \hat{z})/\sqrt{2}$

- Matrix form: $H = |0\rangle\langle+| + |1\rangle\langle-| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
- Note that $H^2 = I$
- Note: Many algorithms start with H, since it maps n qubits initialized in $|0\rangle$ to a superposition of all 2^N orthogonal states
- Identity gate: $I|0\rangle = |0\rangle$ and $I|1\rangle = |1\rangle$

- Leaves state unchanged

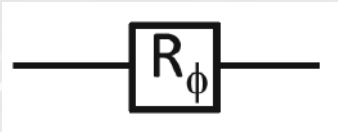
- Matrix form: $I = |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$





Phase gates

- Phase gate: $R_\phi|0\rangle = |0\rangle$ and $R_\phi|1\rangle = e^{i\phi}|1\rangle$

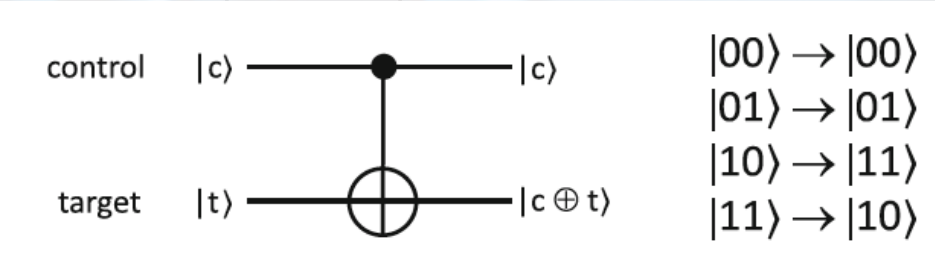


- ϕ rotation of $|1\rangle$ around z-axis

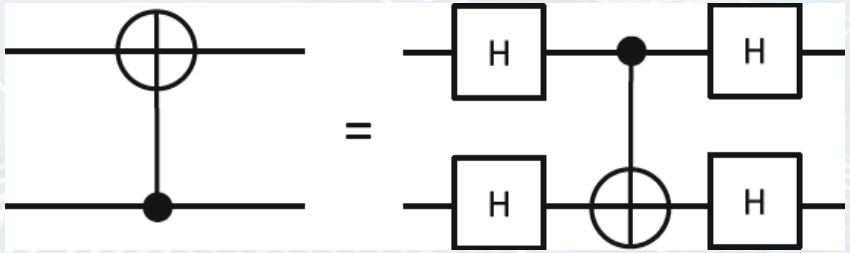
- Matrix form: $R_\phi = |0\rangle\langle 0| + e^{i\phi}|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$
- We can also define:
  - Z gate ($\phi = \pi$): $Z = R_\pi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  - S gate ($\phi = \pi/2$): $S = R_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
  - T gate ($\phi = \pi/4$): $T = R_{\pi/4} = \begin{pmatrix} 1 & 0 \\ 0 & ie^{i\pi/4} \end{pmatrix}$
  - Note that, within a phase factor: $S = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$ and $T = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$
  - (the global phase factor can be neglected)
  - Note: $Z = S^2 = T^4$

2-qubit gates: the CNOT gate

- CNOT flips the second (target) qubit only if the first (control) qubit is $|1\rangle$:



- Input: $|c\rangle|t\rangle$
- Output: $|c\rangle|c \oplus t\rangle$
- Where $\oplus$ is the binary addition (XOR)
- Matrix form: $CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

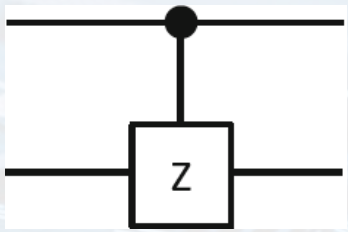


- CNOT can be used to copy a basis state, in fact $CNOT|x\rangle|0\rangle = |x\rangle|x\rangle$
- It does not work with superposition states → no real cloning, in line with the no-cloning theorem

Other 2-qubit gates

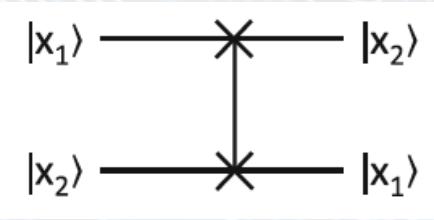
- CZ or CPHASE applies Z to the second qubit only if the first (control) qubit is $|1\rangle$:
- Matrix form: $CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

final |t>	|t>=|0>	|t>=|1>
|c>=$\|0\rangle$	$\|0\rangle$	$\|1\rangle$
|c>=$\|1\rangle$	$\|1\rangle$	$-\|1\rangle$



- SWAP swaps the first and second qubit according to SWAP$|x_1\rangle|x_2\rangle = |x_2\rangle|x_1\rangle$
- Matrix form: $SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

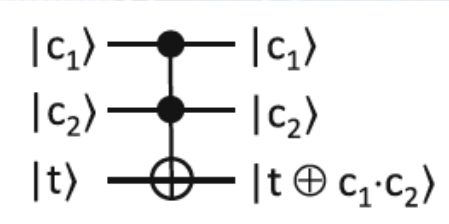
initial	final
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|10\rangle$
$\|10\rangle$	$\|01\rangle$
$\|11\rangle$	$\|11\rangle$



3-qubit gates: the Toffoli gate

- Toffoli or controlled-controlled NOT (CCNOT) is a three qubit gate where the third qubit (target) is flipped only if the first and second qubits are $|1\rangle$
- It can be used to perform AND and NOT classical gates in a reversible way

- Input: $|c_1\rangle|c_2\rangle|t\rangle$
- Output: $|c_1\rangle|c_2\rangle|c \oplus t\rangle$



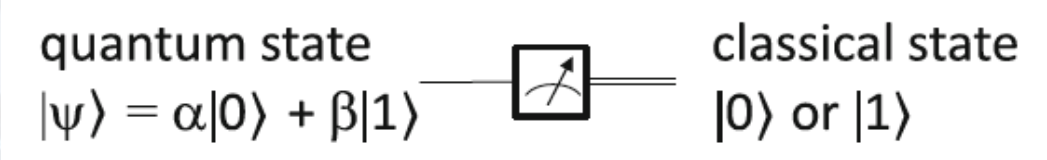
$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Table 7.2 Truth table for the Toffoli gate

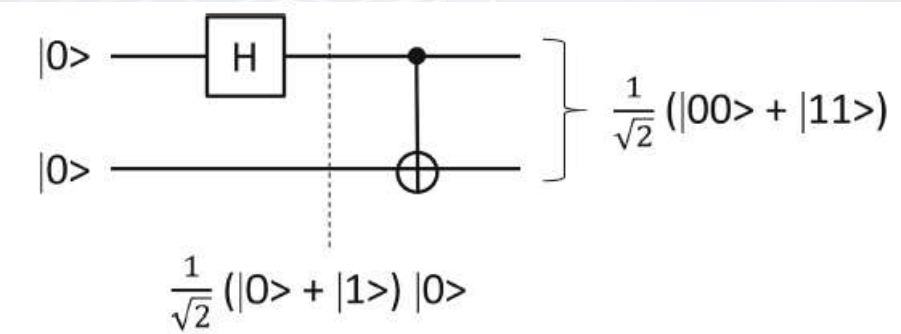
INPUT			OUTPUT		
c_1	c_2	t	c_1	c_2	$t \oplus c_1c_2$
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

Measurement and entanglement

- Measurement causes the collapse of superposition state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ into either $|0\rangle$ or $|1\rangle$ with probabilities $|\alpha|^2$ or $|\beta|^2$ respectively



- Bell circuit consists of Hadamard on the first qubit followed by CNOT, creating a Bell entangled state



- Universal sets for quantum computers:
  - a single-qubit rotation $U(\theta, \phi)$ in the Bloch sphere
  - two-qubit operation such as the controlled-NOT (CNOT) gate

Physical quantum gates

- Time-dependent Schrödinger's equation

$$\hat{H}|\psi\rangle = i\hbar \frac{d}{dt}|\psi\rangle$$

- can be integrated for time-independent $\hat{H}$, to yield:

$$|\psi(t)\rangle = e^{-i(t-t_0)\frac{\hat{H}}{\hbar}}|\psi(t_0)\rangle$$

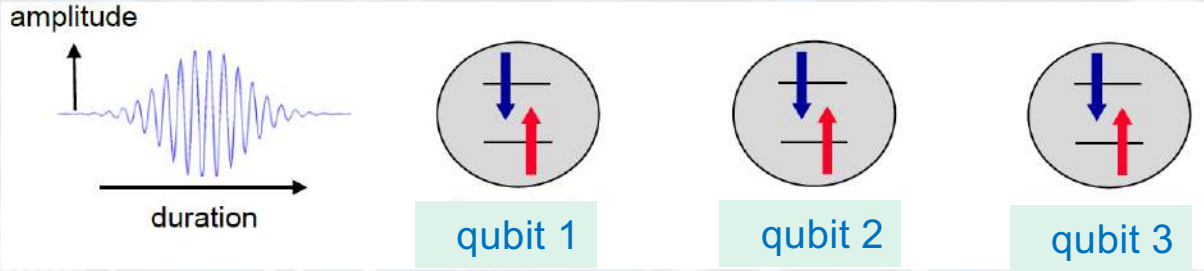
- The quantum gate can be described by the application of an evolution operator $\hat{U}$ for a given time $(t - t_0)$, where $\hat{U}$ is given by:

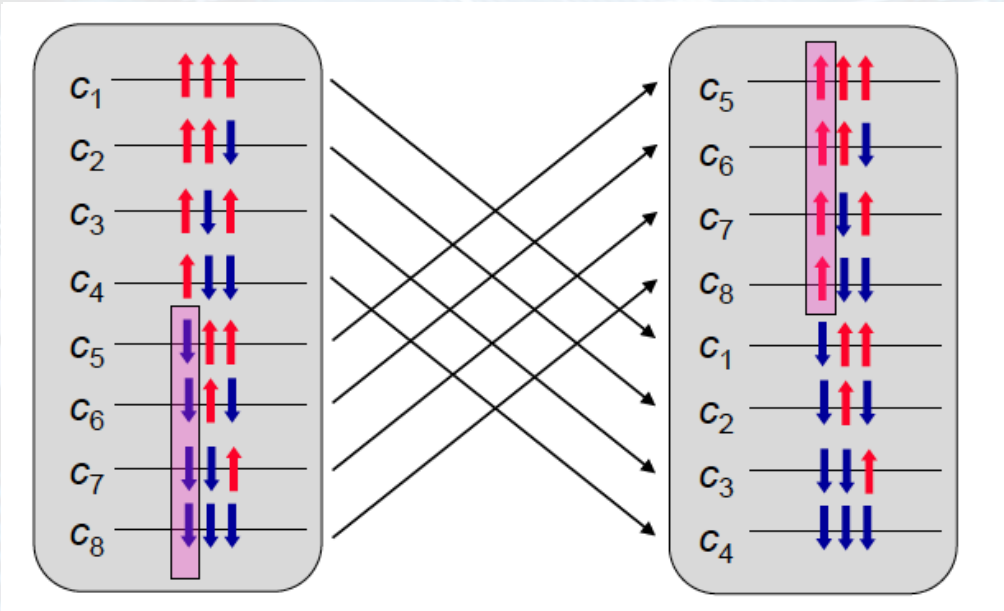
$$\hat{U} = e^{-i(t-t_0)\frac{\hat{H}}{\hbar}}$$

- Generally, quantum gates (e.g., Pauli gates) are obtained by applying high frequency electromagnetic pulses via electron spin resonance (ESR)

Quantum parallelism

- EM pulse flips qubit 1 (X or NOT gate)





W. D. Oliver, ISSCC Tutorial (2023)

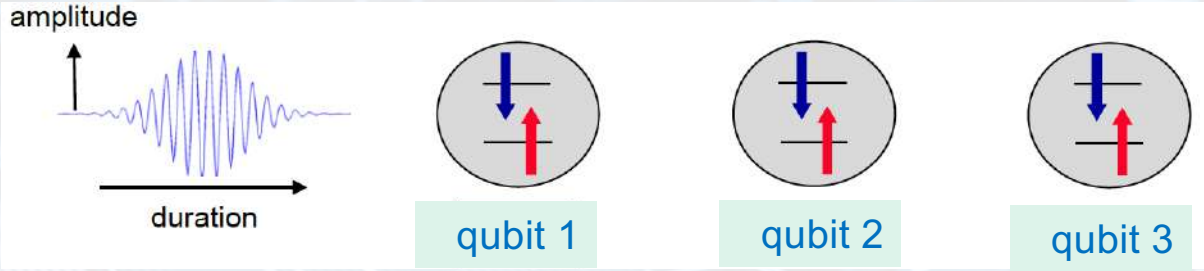
- Quantum gate operates on a single qubit however the effects propagate on the entire system simultaneously → quantum parallelism

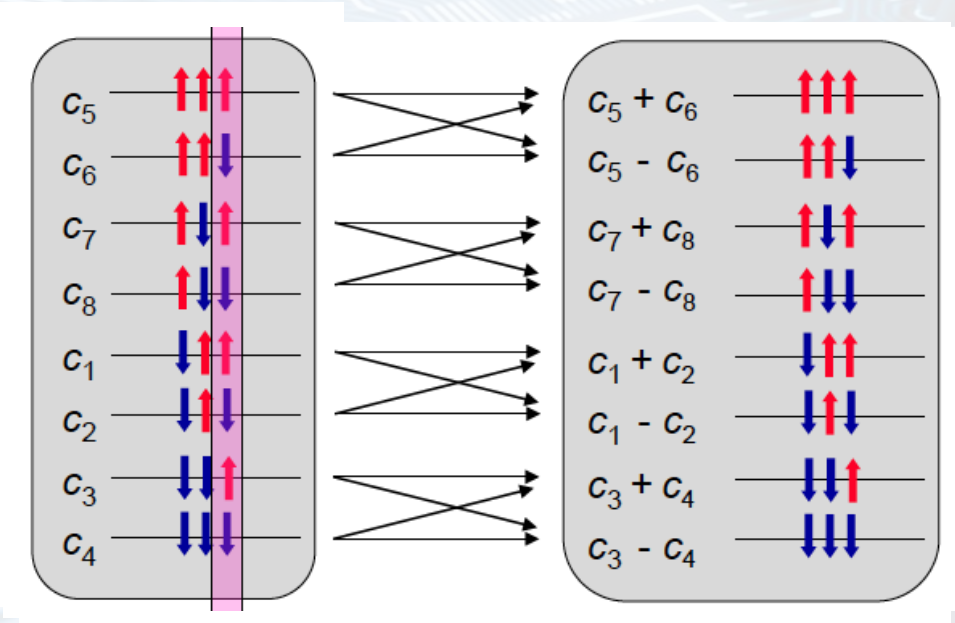
Quantum interference

- EM pulse rotates qubit 3 by $\pi/2$ (H gate)

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$





Constructive interference

Destructive interference

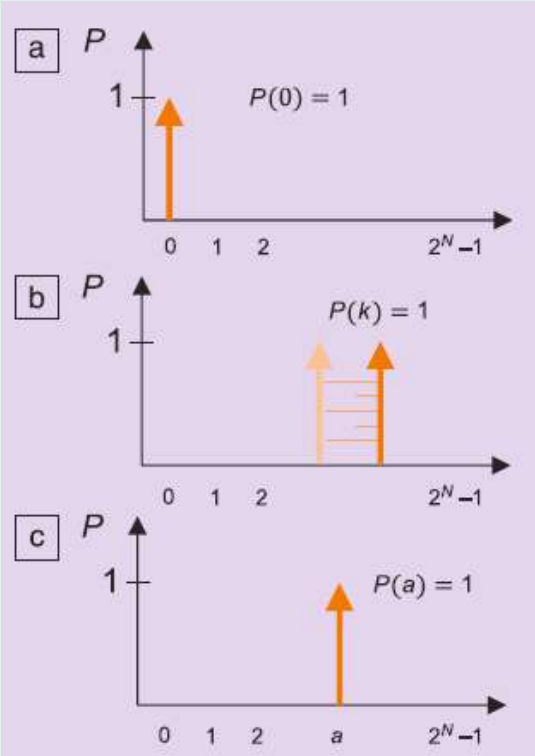
W. D. Oliver, ISSCC Tutorial (2023)

- Quantum gate can lead to quantum interference with quantum parallelism

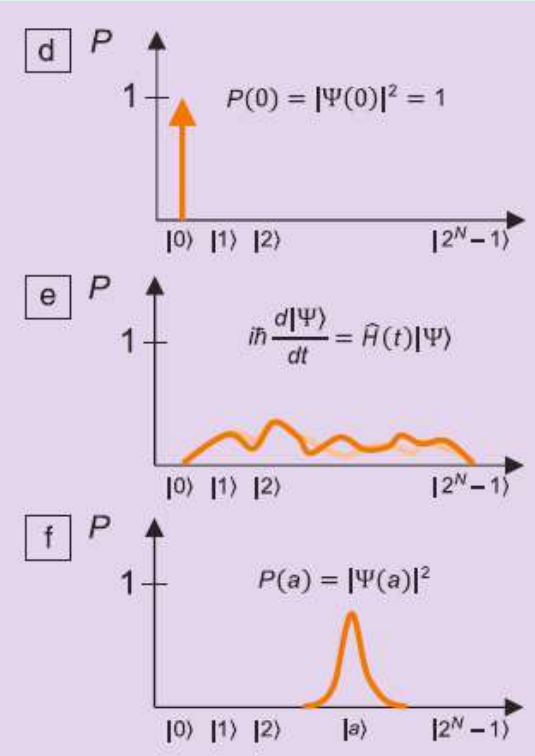
Sommario

- Introduzione
- Quantum bits
- Quantum gates
- Quantum circuits
- Conclusioni

How does a quantum computer compute?



- Initialization: all bits prepared in state 0
- Computation: Boolean interaction moves bits, every time sitting in just one state
- Read: final bit states are finally measured to learn the results



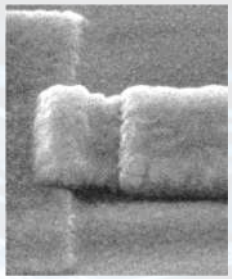
- Initialization: all qubits prepared in reference state $|\psi\rangle = |00 \ldots 00\rangle$
- Computation: wavefunction $|\psi\rangle$ evolves according to the Schrödinger's equation, every time sitting in all states
- Read: final qubit state $|\psi\rangle$ is measured by projecting it along a certain basis state $|a\rangle$

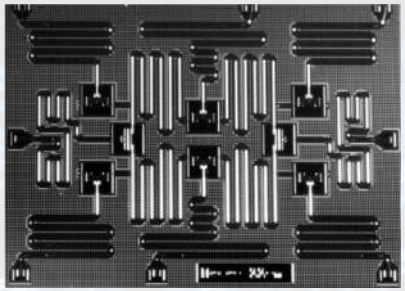
- Finally, only basis state $|a\rangle$ should have a coefficient close to 1, all others should be close to 0, so that the measurement yields $|a\rangle$ i.e. the solution to the problem

J. N. Eckstein, et al., MRS Bull. 38, 783 (2013)

DiVincenzo criteria for quantum computing

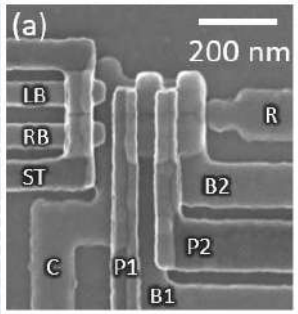
1. Robust, manufacturable qubit technology





2. Initialization to a basis state |000…0>

3. Universal set of quantum gates (e.g., H, S, T and CNOT)



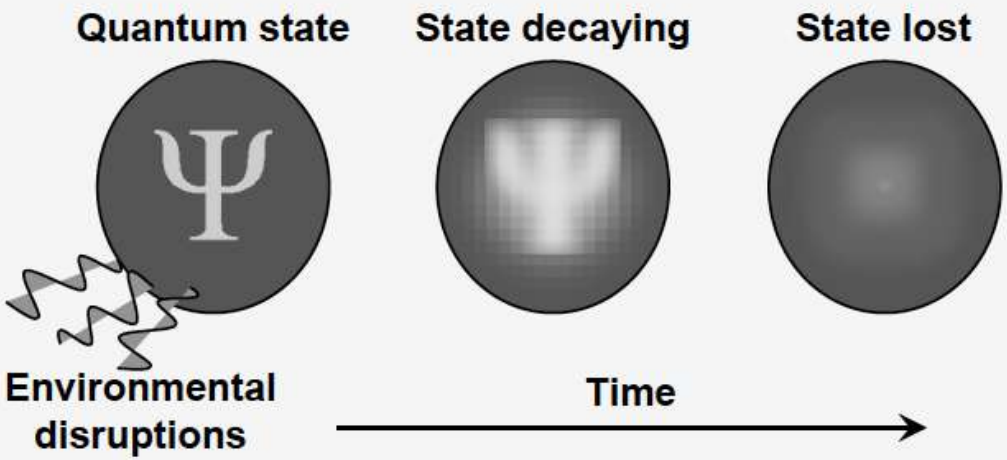
4. Qubit-specific measurement

5. Long coherence $T_{coh}/T_{gate} > 10^4$

D. P. DiVincenzo, 'Topics in Quantum Computers,' Mesoscopic Electron Transport. 345 (1997)

Coherence time

Coherence time t_{coh}: The qubit's lifetime



Gate time t_{gate}: Time required for a single gate operation

Figure of Merit * : # of gates per coherence time = t_{coh}/t_{gate}

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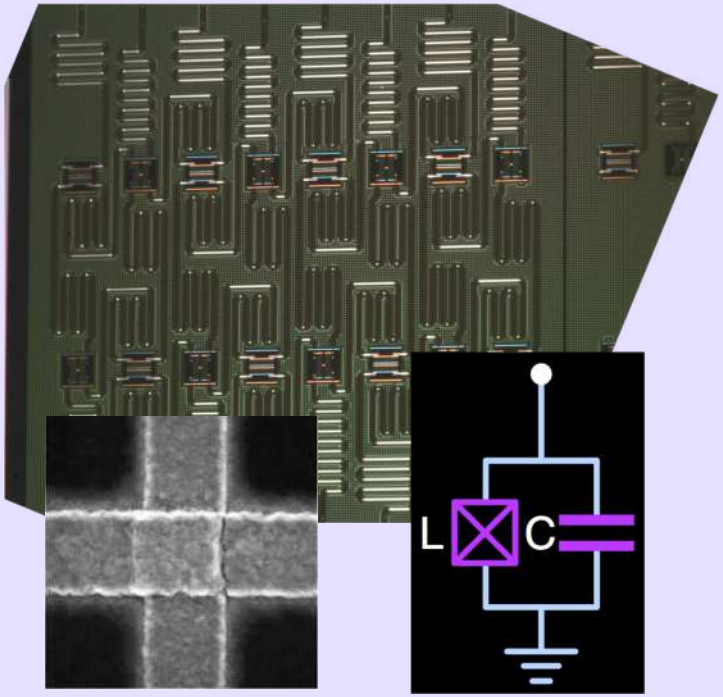
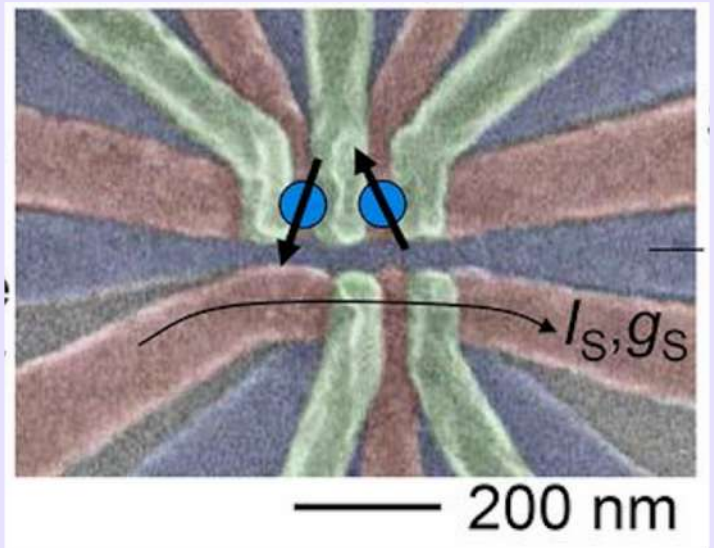
Quantum hardware

Superconducting qubit

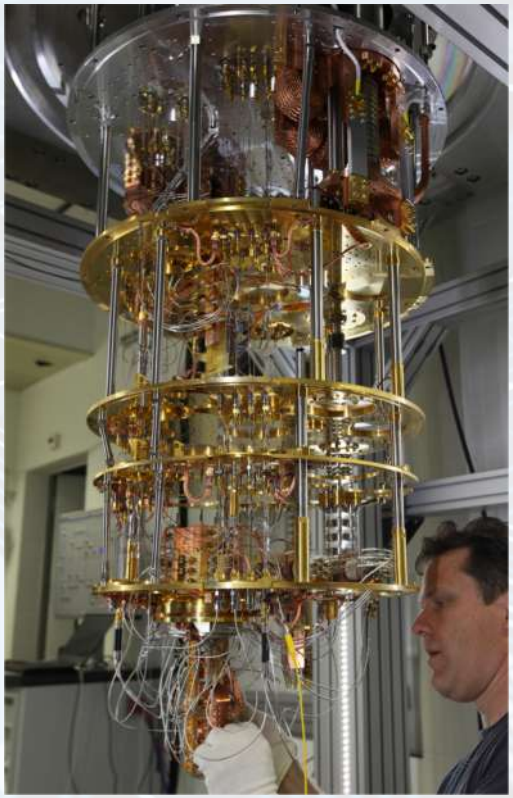


Oscillatore LC quantistico con una giunzione Josephson (Google, IBM)

Quantum dot qubit

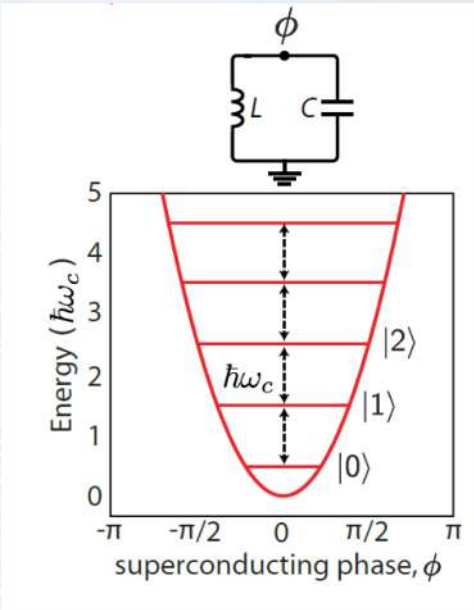


Isole quantiche di silicio dove il qubit è dato dallo spin (momento magnetico) dell'elettrone (Intel)



Superconducting qubit: physics

- LC oscillator includes 2 types of energy:
  - Electrical energy (charge) $E_C = \frac{Q^2}{2C}$ (1)
  - Magnetic energy (flux) $E_\phi = \frac{\phi^2}{2L}$ (2)
- Energy oscillates between L and C, same as an electromagnetic (harmonic) oscillator
- The quantum system is described by Hamiltonian: $\mathrm{H} = \frac{Q^2}{2C} + \frac{\phi^2}{2L}$
- Similar to: $\mathrm{H} = \frac{p^2}{2m} + k\frac{x^2}{2}$
- with kinetic energy (1), potential energy (2) and eigenvalues given by:
- $E = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right)$ where $\omega = \frac{1}{\sqrt{LC}}$ (3)
- The equal spacing of eigenvalues is an issue with quantum bit operation relying on only two basis states $|0\rangle$ and $|1\rangle$

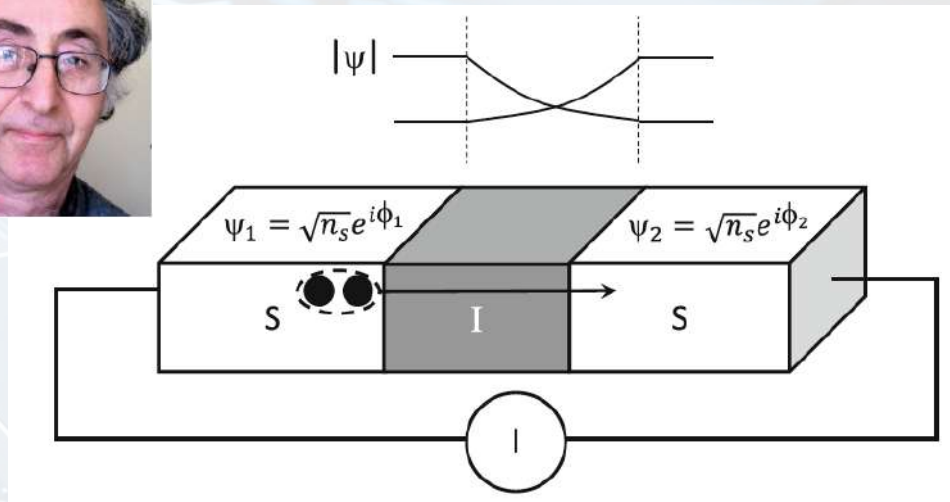


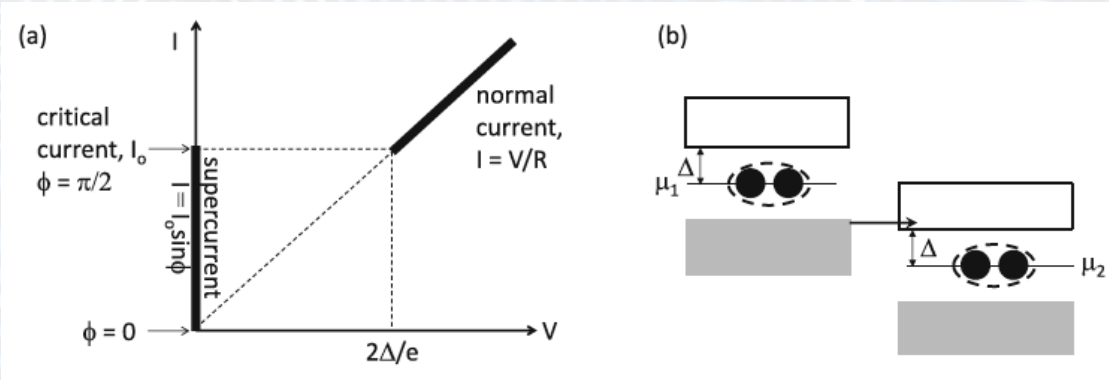
The Josephson junction (JJ)

- JJ is a superconducting tunnel junction (e.g., $Al/Al_2O_3/Al$ below the critical temperature of 280 mK) based on coherent tunneling of Cooper pairs
- The tunneling current is given by:

$$I = I_0 \sin\phi$$

- where I_0 is the critical current and $\phi = \phi_1 - \phi_2$ is the phase difference across the junction with $V = 0$
- Above I_0, the JJ transitions to a ohmic behavior due to direct tunneling of individual electrons above the superconducting band gap of energy 2Δ



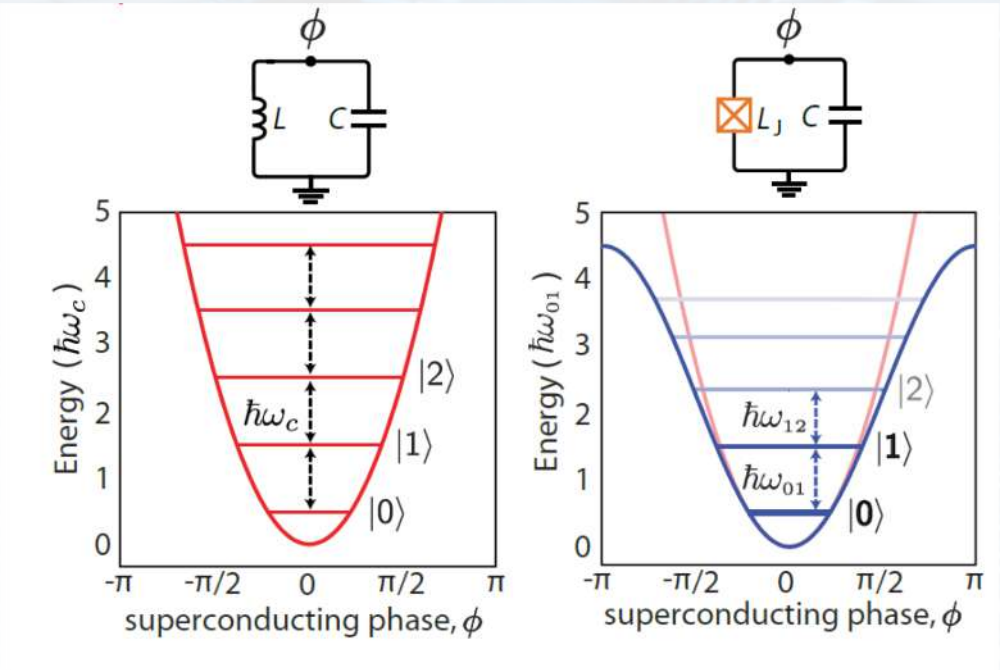


Anharmonic oscillator

- Replacing the inductance L with a JJ leads to a new Hamiltonian:

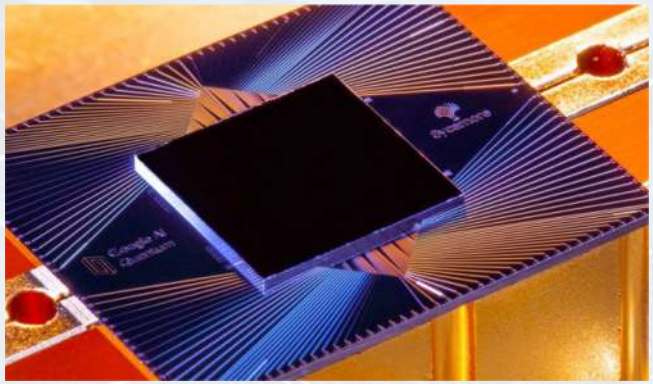
$$\mathrm{H} = -4E_C \frac{\partial^2}{\partial\phi^2} - E_J cos\phi$$

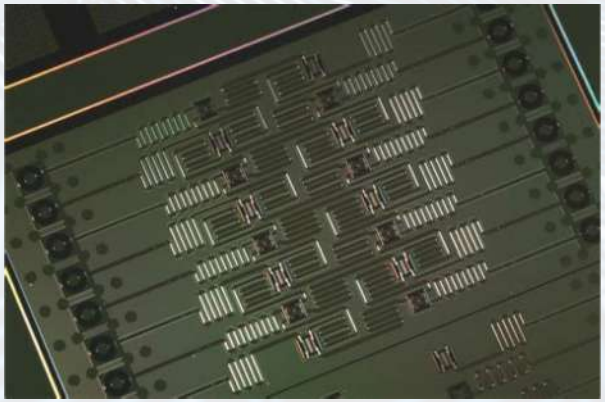
- Which breaks the degeneracy of the eigenvalues → control of qubit thanks to the unique resonance energy / Rabi frequency
- The typical resonance frequency is 5 GHz → the qubit must be operated at temperature T<hν/k = 240 mK



Superconducting qubit: state of the art

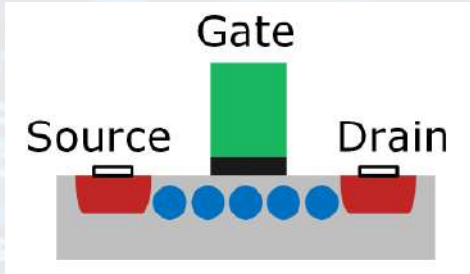
- Coherence times: ~ 100 μs
- Fidelity and operation times
  - 1 QB: 99.99% in 10 ns
  - 2 QB: 99.9% in 40 ns
  - Readout: 99.0% in 200 ns
- Clock rate: ~ 25 MHz
- Largest algorithm: 53 qubits
- Companies:
  - AWS, Google, IBM, QCI, Rigetti
  - Annealing: D-Wave

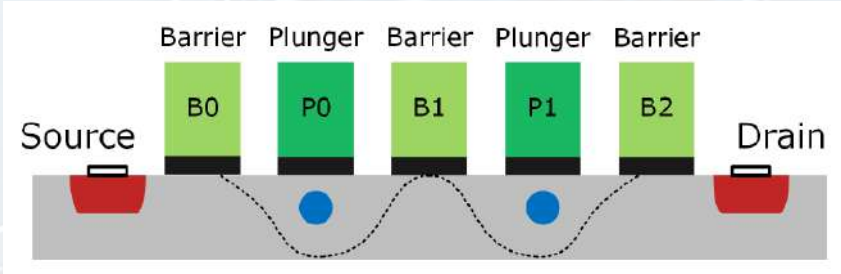




Spin qubit: physics

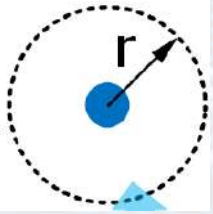
- A multigate transistor with 'barrier' gates and 'plunger' gates
- Quantum dots (QD) are formed electrostatically
- Single electrons are trapped and their spin is used as basis state for the qubit
- QD capacitance C = $4\pi\varepsilon r$
- Charging energy $E_C = \frac{e^2}{8\pi\varepsilon r}$
- R = 100 nm → E_C = 1 meV and T = E_C/k = 10 K





$$C = 4\pi\epsilon r$$

$$E_C = \frac{e^2}{8\pi\epsilon r}$$



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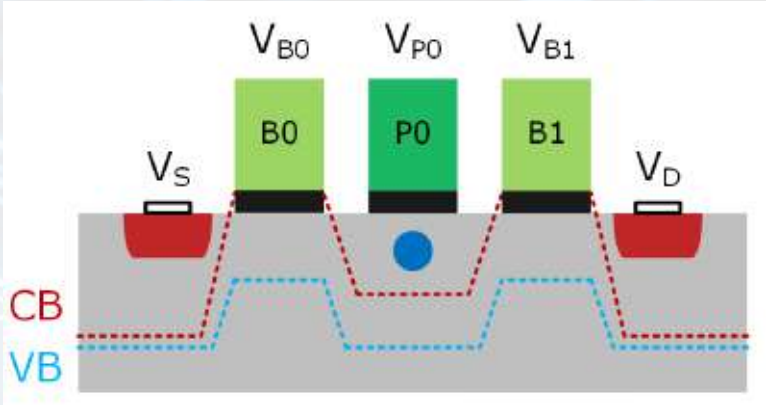
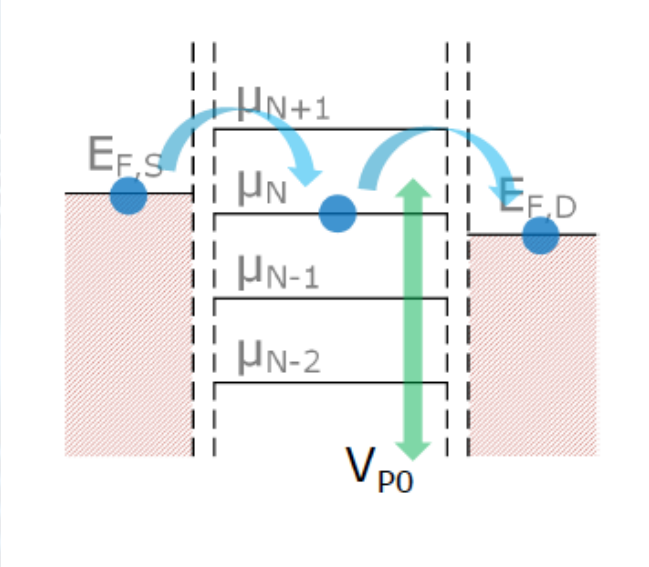
Coulomb blockade

- For each added electron, the energy increases by E_C → energy ladder for increasing number of trapped electrons N
- For small V_{DS}, no current can flow if $E_{F,S}$ and $E_{F,D}$ are between μ_{N-1} and μ_N → Coulomb blockade (classical, not quantum effect)
- The plunger gate voltage V_{P0} can be changed to align μ_N between $E_{F,S}$ and $E_{F,D}$ → current can flow by one electron tunneling in, followed by one electron tunneling out
- Coulomb blockade can be used to charge and trap electrons in the QD





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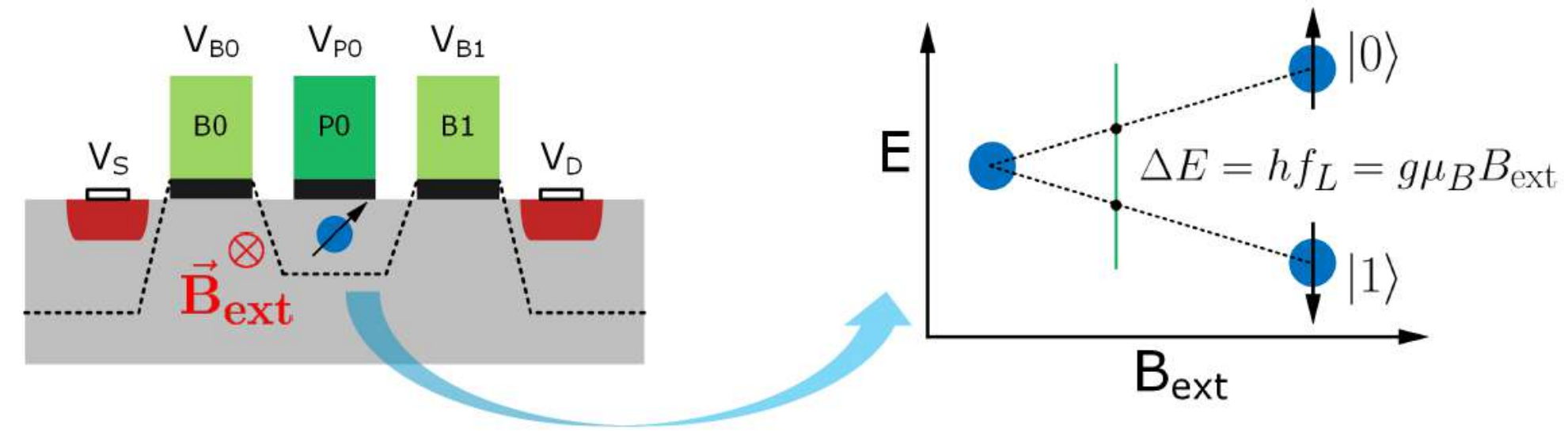
Spin states by Zeeman splitting



- Applied B_{ext} splits the energies of spin up and down states
- Assuming g = 2, B_{ext} = 0.5 T → ΔE = 58 μeV, f_L = 14 GHz, T = 670 mK
- Qubit manipulation by ESR, qubit readout by spin-charge conversion

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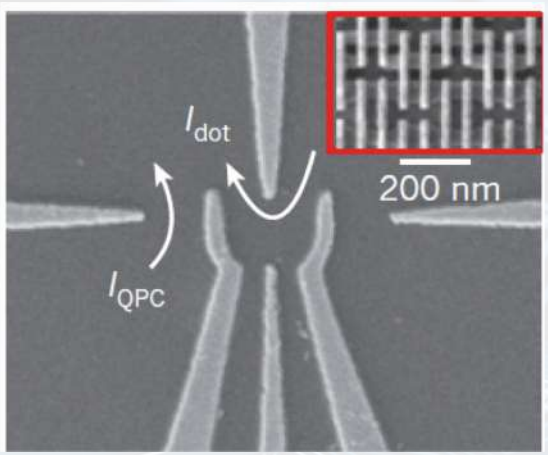
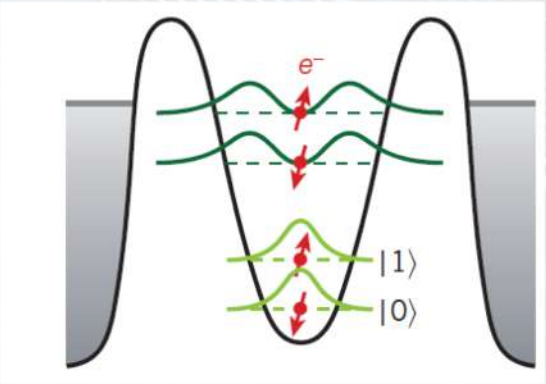
Spin qubit

- Coherence time = 400 μs
- Fidelity and gate time
  - 1QB: 99.5%, 100 ns
  - 2QB: 99%, 200 ns
- Clock rate = 5 MHz
- Largest algorithm = 6 qubits
- Companies: Intel, …





Sommario

- Introduzione
- Quantum bits
- Quantum gates
- Quantum circuits
- Conclusioni

Conclusioni

- Dal punto di vista didattico, il quantum computing abbraccia tematiche che vanno dalla fisica dei dispositivi alla progettazione di circuiti integrati per la manipolazione (quantum gate) e la lettura (readout) dei qubit
- Le sfide sono enormi: aumentare la quantum fidelity mediante la correzione dell'errore, aumentare il numero di qubit, integrare il sistema su un singolo chip
- Sebbene non esista ancora un mercato ed una tecnologia consolidata, è utile conoscere i principi base e i vantaggi/svantaggi del quantum computing